

Indirect Coupling of the Cell Method and BEM for Solving Three-Dimensional Unbounded Magnetostatic Problems

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A novel hybrid approach for solving magnetostatic problems with an unbounded air domain is presented. The basic idea is to use augmented dual grids for interfacing the Cell Method and BEM by indirect coupling, introducing equivalent surface field sources. The field problem in finite domains is discretized by the Cell Method in terms of integral variables, i.e. line integrals of the magnetic vector potential. Boundary integral conditions, formulated with the reduced magnetic scalar potential, are applied to avoid the air region meshing. A mixed final symmetric algebraic system, which can be solved by fast iterative solvers like MINRES or SYMMLQ, is finally obtained. The magnetic field in the air region is then easily reconstructed from equivalent sources. Numerical tests show that the hybrid method is accurate even by using a collocation approach for discretizing boundary integral conditions.

Index Terms—Cell Method, Finite Integration Technique, Boundary Element, Integral Equation, Magnetostatics.

I. INTRODUCTION

HYBRID methods combining FEM and BEM advantages have been developed in computational electromagnetics since the early '80s [1][2]. Main key strengths are avoiding air region meshing and allowing easier pre- and post-processing compared to FEM. These basic features generally lead to time and cost reduction in the design of new products [3].

In the so-called *discrete approaches*, like the Cell Method (CM) or the Finite Integration Technique (FIT), field problems are formulated directly in algebraic form by using integral variables, e.g. line integrals and fluxes. A CM-based integral formulation for eddy current problems in thin shells was presented in [4] and then extended to multiply connected shells in [5]. It was proved that the CM can be coupled with the BEM for magnetostatics [6] and for magnetodynamics [7]. Unfortunately, both of these methods led to unsymmetric final system matrices solved by (LU) direct or iterative (GMRES) solvers, which are typically resource and time consuming.

Recently, it was shown in [8] that boundary conditions in both CM and FIT can be discretized by introducing a pair of *augmented dual grids*. Starting from this result, a novel hybrid approach for coupling CM and BEM in three-dimensional magnetostatic problems is here presented. This leads to a final algebraic mixed system matrix that is symmetric so that fast iterative solvers like MINRES or SYMMLQ can be used.

II. HYBRID FORMULATION

Let $\Omega = \bigcup_k \Omega_k$ be the *interior region*, i.e. the the union of n finite and connected subdomains $\Omega_k \subset \mathbb{R}^3$, $k = 1 \dots n$, where magnetic materials are contained. Let $\Omega^C = \mathbb{R}^3 \setminus \Omega$ be the *exterior region*, i.e. the complementary part of Ω in the free space, which is unbounded and includes magnetic field sources. The interface boundary between interior and exterior regions is then defined as $\Gamma = \bigcup_k \Gamma_k$, where $\Gamma_k = \partial\Omega_k$ is the boundary of each subdomain Ω_k and is connected.

A. Interior Problem

As shown in [8], any subdomain has to be meshed by augmented dual grids in order to properly discretize boundary conditions by the CM. These grids are constructed as follows. Dual grids \mathcal{G}_Ω and \mathcal{G}_Γ are first defined on Ω and Γ by joining centroids of oriented edges, faces, and volumes of the primal grid \mathcal{G}_Ω (tetrahedral mesh) and \mathcal{G}_Γ (the restriction of \mathcal{G}_Ω on Γ). The augmented dual grid is then obtained as the union $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_\Omega \cup \tilde{\mathcal{G}}_\Gamma$. Dual grids are defined along with the following incidence matrices: \mathbf{C}_Ω (faces to edges on \mathcal{G}_Ω), $\tilde{\mathbf{C}}_\Omega = \mathbf{C}_\Omega^T$ (faces to edges on $\tilde{\mathcal{G}}_\Omega$), \mathbf{C}_Γ (faces to edges on \mathcal{G}_Γ), $\tilde{\mathbf{G}}_\Gamma = \mathbf{C}_\Gamma^T$ (edges to nodes on $\tilde{\mathcal{G}}_\Gamma$), $\tilde{\mathbf{C}}_{\Omega\Gamma}$ (faces on $\tilde{\mathcal{G}}$ to edges on $\tilde{\mathcal{G}}_\Gamma$).

Field problem variables of CM for magnetostatics are arrays of DoF on \mathcal{G}_Ω and $\tilde{\mathcal{G}}$: $\mathbf{a}_\Omega, \mathbf{a}_\Gamma$ (magnetic vector potential line integrals on primal edges), $\mathbf{b}_\Omega, \mathbf{b}_\Gamma$ (magnetic fluxes on primal faces), $\tilde{\mathbf{h}}_\Omega, \tilde{\mathbf{h}}_\Gamma$ (magnetic field line integrals on dual edges).

Topological equations are defined on each dual grid apart. Magnetic fluxes on \mathcal{G}_Ω fulfill Gauss' law $\mathbf{b}_\Omega = \mathbf{C}_\Omega \mathbf{a}_\Omega$, for simply connected domains, and line integrals of the magnetic field on $\tilde{\mathcal{G}}$ fulfill Amperé's law $\tilde{\mathbf{C}}_\Omega \tilde{\mathbf{h}}_\Omega + \tilde{\mathbf{C}}_{\Omega\Gamma} \tilde{\mathbf{h}}_\Gamma = 0$. Electric currents are not here considered, since field sources are in Ω^C .

The matrix constitutive relation for magnetic linear media is here obtained by using the so-called *energetic approach*, presented in [9]. The local relationship $\mathbf{H}(x) = \nu \mathbf{B}(x)$, where ν is the magnetic reluctivity, is expanded by piecewise uniform bases \mathbf{w}_f defined for any primal face f and the corresponding global relationship at discrete level becomes: $\tilde{\mathbf{h}}_\Omega = \mathbf{M}_\nu \mathbf{b}_\Omega$, where matrix coefficients are $\mathbf{M}_{\nu,ij} = \int_\Omega \nu \mathbf{w}_i(x) \cdot \mathbf{w}_j(x) dx$.

Combining topological and constitutive relationships the global equation for the interior magnetic problem in Ω reads:

$$\tilde{\mathbf{C}}_\Omega \mathbf{M}_\nu \mathbf{C}_\Omega \mathbf{a}_\Omega + \tilde{\mathbf{C}}_{\Omega\Gamma} \tilde{\mathbf{h}}_\Gamma = \mathbf{0} \quad (1)$$

where the array $\tilde{\mathbf{h}}_\Gamma$ is used for the interface coupling.

B. Exterior Problem

Boundary integral conditions for the interior problem are derived by an indirect BEM approach [2], defined as follows.

The source magnetic field is given by Biot-Savart's integral:

$$\mathbf{H}_0(x) = 1/4\pi \int_{\Omega_0} \mathbf{J}(y) \times \frac{x-y}{\|x-y\|^3} dy \quad (2)$$

Because $\nabla \cdot \mathbf{H}_0 = 0$ and by the Helmholtz theorem, the magnetic field in Ω^C splits as $\mathbf{H} = \mathbf{H}_r + \mathbf{H}_0$, where $\mathbf{H}_r = -\nabla\varphi$ is the reduced field. The scalar potential φ is found after solving an *Exterior Neumann Problem*, with $\partial_n \varphi = g$ as (unknown) boundary condition [10]. If $\int_{\Gamma} g(x) dx = 0$, it is shown that Fredholm's integral equation:

$$-1/2 \sigma(x) + \mathcal{T}^*[\sigma](x) = g(x), \quad x \in \Gamma \quad (3)$$

admits a unique solution σ such that

$$\varphi(x) = K[\sigma](x) = \int_{\Gamma} \Phi(x,y) \sigma(y) dy, \quad x \in \Omega^C \quad (4)$$

where $\Phi(x,y) = 1/(4\pi)\|x-y\|^{-1}$ is the fundamental solution for Δ and $\mathcal{T}^*[\sigma](x) = \int_{\Gamma} \sigma(y) \partial_n \Phi(x,y) dy$, $x \in \Gamma$.

The *Steklov-Poincaré operator* S , mapping the Dirichlet into the Neumann datum, is obtained from (3) and (4). By inverting (4) and by letting σ in (3), it can be expressed as:

$$g(x) = S[\varphi](x) = (-1/2 + \mathcal{T}^*) \circ K^{-1}[\varphi](x) \quad (5)$$

for any $x \in \Gamma$. This operator can be discretized by collocation or Galerkin methods as $\mathbf{b}_{r,\Gamma} = \mathbf{S} \tilde{\varphi}_{\Gamma}$, where $\mathbf{b}_{r,\Gamma}$ is the array of the reduced magnetic flux density and $\tilde{\varphi}_{\Gamma}$ is the array of scalar potentials on $\tilde{\mathcal{G}}_{\Gamma}$. \mathbf{S} is made symmetric as in [1].

C. Coupled system

The interior and exterior problems are coupled by the following interface conditions: the continuity of the magnetic field line integrals $\tilde{\mathbf{h}}_{\Gamma} = \tilde{\mathbf{h}}_{0,\Gamma} + \tilde{\mathbf{G}}_{\Gamma} \varphi_{\Gamma}$ and the continuity of magnetic fluxes $\mathbf{b}_{\Gamma} = \mathbf{b}_{0,\Gamma} + \mathbf{b}_{r,\Gamma}$, where subscript 0 indicates arrays due to source fields. Combining interface conditions and equations (1), a symmetric indefinite system is obtained:

$$\begin{pmatrix} \mathbf{C}_{\Omega}^T \mathbf{M}_{\nu} \mathbf{C}_{\Omega} & \tilde{\mathbf{C}}_{\Omega\Gamma} \tilde{\mathbf{G}}_{\Gamma} \mathbf{P}_{\Gamma} \\ \mathbf{P}_{\Gamma}^T \tilde{\mathbf{G}}_{\Gamma}^T \tilde{\mathbf{C}}_{\Omega\Gamma}^T & -\mathbf{P}_{\Gamma}^T \mathbf{S} \mathbf{P}_{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{\Omega} \\ \varphi_{\Gamma} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathbf{C}}_{\Omega\Gamma} \tilde{\mathbf{h}}_{0,\Gamma} \\ \mathbf{P}_{\Gamma}^T \mathbf{b}_{0,\Gamma} \end{pmatrix} \quad (6)$$

where matrix \mathbf{P}_{Γ} projects nodal DoF from \mathcal{G}_{Γ} to $\tilde{\mathcal{G}}_{\Gamma}$. Note that, due to system symmetry, efficient iterative solvers like MINRES or SYMMLQ can be used.

The equivalent source distribution $\sigma(x)$ on the discretized boundary is reconstructed from φ_{Γ} after solving the final system (6). The magnetic field in Ω^C is finally obtained at the post-processing stage, as:

$$\mathbf{H}(x) = \mathbf{H}_0(x) - \int_{\Gamma} \nabla_x \Phi(x,y) \sigma(y) dy \quad (7)$$

without need of numerical differentiation as with direct BEM, where the scalar potential (4) has to be explicitly computed.

III. NUMERICAL RESULTS

A 2D axisymmetric magnetostatic problem is considered in order to get highly accurate results from 2D FEM simulations. Let (r, z) be cylindrical coordinates. A magnetic disc (relative magnetic permeability $\mu_r=1,000$, 1 m radius, 0.2 m thickness), centered at the origin, is excited by a circular loop (100 A current, 0.4 m radius), centered at (0,0.4). The disc is discretized by 28,035 tetrahedral and 4,280 boundary triangular elements. The FEM model is embedded into a disc (5 m radius) with infinite BCs, discretized by 11,440 3rd order triangles.

The magnetic flux density components B_r, B_z are compared in Fig. 1 on a horizontal line ($x = [0, 1], y = 0, z = 0.2$), located above the disc. Three-dimensional Hybrid Cell Method (3D HCM) shows to be in very good agreement with 2D FEM. The maximum discrepancies are $e_{B_r} = 1.14\%$, $e_{B_z} = 0.54\%$.

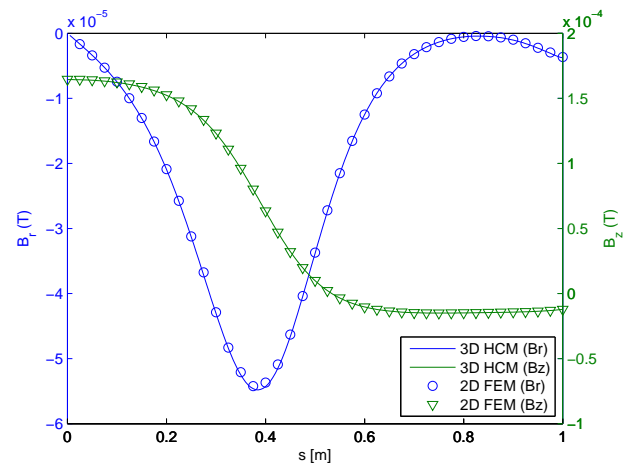


Fig. 1. Magnetic flux density components B_r (blue line) and B_z (green line) for HCM and FEM, computed along the line $x=[0,1], y=0, z=0.2$.

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